

# The Schrödinger-Virasoro Lie algebra: a mathematical structure between conformal field theory and non-equilibrium dynamics

Jérémie Unterberger

Institut Elie Cartan, <sup>1</sup> Université Henri Poincaré Nancy I, B.P. 239, F-54506  
Vandœuvre-lès-Nancy Cedex, France

E-mail: [jeremie.unterberger@iecn.u-nancy.fr](mailto:jeremie.unterberger@iecn.u-nancy.fr)

**Abstract.** We explore the mathematical structure of the infinite-dimensional Schrödinger-Virasoro algebra, and discuss possible applications to the integrability of anisotropic or out-of-equilibrium statistical systems with a dynamical exponent  $z \neq 1$  by defining several correspondences with conformal field theory.

**0.** Two-dimensional conformal field theory is an attempt at understanding the universal behaviour of two-dimensional statistical systems at equilibrium and at the critical temperature, where they undergo a second-order transition. Starting from the basic assumption of translational and rotational invariance, together with the fundamental hypothesis that scale invariance holds at criticality, one is naturally led to the idea (for systems with short-ranged interactions) that invariance under the whole conformal group should also hold. Local conformal transformations in *two* dimensions are generated by infinitesimal holomorphic or anti-holomorphic transformations which make up two copies of the celebrated Virasoro algebra  $\mathfrak{vir}^k \cong \langle L_n \mid n \in \mathbb{Z} \rangle$ , with Lie brackets  $[L_n, L_m] = (n - m)L_{n+m} + kc(L_n, L_m)$ , where  $c(L_n, L_m) = \frac{1}{12}n(n^2 - 1)\delta_{n+m,0}$  is the *Virasoro cocycle*, defining a *central extension*<sup>2</sup> of  $\mathfrak{vir}^0$ . A systematic investigation of the theory of representations of the Virasoro algebra in the 80's led to introduce a class of physical models, called *unitary minimal models*, corresponding to the unitary highest weight representations of  $\mathfrak{vir}$  with central charge less than 1, whose correlators are entirely determined by the symmetries (see for instance [2]).

Nothing of the sort exists for the time being in non-equilibrium or anisotropic statistical physics. The natural starting point for mathematical investigations is the scaling hypothesis, which states that correlators should be invariant under time and space translations and under *anisotropic dilations*  $\vec{r} \rightarrow \lambda \vec{r}$ ,  $t \rightarrow \lambda^z t$ , where  $\vec{r}$  and  $t$  stand for 'space' and 'time' coordinates, and  $z$  is called the *anisotropy* or *dynamical exponent*. This applies to critical dynamics (see [4],[11]), phase ordering kinetics (see [1]) and also to problems of statistical mechanics with a strong anisotropy, such as directed percolation [10] or Lifchitz points [13].

The case  $z = 2$  is quite naturally associated with Galilei invariance or full Schrödinger invariance, see (4) below. For instance, dynamical systems with  $z = 2$  are often described in

<sup>1</sup> Laboratoire associé au CNRS UMR 7502

<sup>2</sup> Central extensions describe important fluctuation effects in field theories, e.g. the Casimir effect.

terms of a stochastic Langevin equation

$$(2\mathcal{M}\partial_t - \partial_r^2)\Phi(t, r) = F(\Phi) + \eta \quad (1)$$

where  $\eta$  is a Gaussian white noise. Recently it was pointed out that *if* the deterministic part of (1) is Galilei invariant, the noise-averaged  $n$ -point correlation and response functions can be rewritten in terms of certain correlators of the noiseless theory (where  $\eta = 0$ ), see [14]. Therefore, one is interested in the symmetries of non-linear PDE's whose archetypes are diffusion equations of the type

$$2\mathcal{M}\partial_t\Phi - \partial_r^2\Phi = F \quad (2)$$

where  $F = F(t, r, \Phi, \Phi^*)$  is a potential. This set of admissible equations may be enlarged by taking a Fourier transform with respect to the mass,  $\tilde{\Phi}(\zeta) = \int_{\mathbb{R}} \Phi \mathcal{M} e^{-i\mathcal{M}\zeta} d\mathcal{M}$ , yielding equations of the type

$$(2\partial_\zeta\partial_t - \partial_r^2)\tilde{\Phi}(t, r, \zeta) = F(t, r, \zeta, \tilde{\Phi}, \tilde{\Phi}^*) \quad (3)$$

as well as by considering a potential depending on a dimensionful coupling constant  $g$  (see [16]). In any case, the largest possible group of invariance (as obtained for  $F \equiv 0$ ), say in one space dimension for simplicity, is the Schrödinger group  $\mathfrak{sch}_1$  with Lie algebra

$$\mathfrak{sch}_1 \cong \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{gal} = \langle L_{-1}, L_0, L_1 \rangle \ltimes \langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0 \rangle, \quad (4)$$

with  $L_{-1} = -\partial_t$ ,  $Y_{-\frac{1}{2}} = -\partial_r$  (translations),  $L_0 = -t\partial_t - \frac{1}{2}r\partial_r$  (infinitesimal anisotropic dilation for  $z = 2$ ),  $Y_{\frac{1}{2}} = -t\partial_r - \mathcal{M}r$  (Galilei transformation, including a phase term  $\mathcal{M}r$  multiplying the wave function  $\Phi$ ) and  $M_0 = -\mathcal{M}$ , which exhibits a semi-direct product structure; in other words,  $\mathfrak{gal}$  is both a subalgebra of  $\mathfrak{sch}_1$  and (as a vector space) a representation of  $\mathfrak{sl}(2, \mathbb{R})$ . So, even for one space dimension, there is *seemingly* no analogue of the infinite-dimensional conformal group of conformal field theory.

1. Yet –in some sense– there *are* several substitutes for  $\mathfrak{vir}^k$  in this setting. We shall mainly concentrate on the so-called *Schrödinger-Virasoro Lie algebra*, first introduced in [6]

$$\mathfrak{sv} = \langle L_n, Y_m, M_p \mid n, p \in \mathbb{Z}, m \in \mathbb{Z} + \frac{1}{2} \rangle \quad (5)$$

with Lie brackets

$$\begin{aligned} [L_n, L_p] &= (n-p)L_{n+p}, \quad [L_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m}, \quad [L_n, M_p] = -pM_{n+m} \\ [Y_m, Y_{m'}] &= (m - m')M_{m+m'}, \quad [Y_m, M_p] = 0, \quad [M_n, M_p] = 0 \end{aligned} \quad (6)$$

where  $n, p \in \mathbb{Z}, m, m' \in \frac{1}{2} + \mathbb{Z}$ . It is an infinite-dimensional 'extension' of  $\mathfrak{sch}_1$ , with a semi-direct product structure  $\mathfrak{sv} \cong \mathfrak{g} \ltimes \mathfrak{h}$  – where  $\mathfrak{g} = \langle L_n \mid n \in \mathbb{Z} \rangle \cong \mathfrak{vir}^0$  and  $\mathfrak{h} = \langle Y_m, M_p \mid m \in \mathbb{Z} + \frac{1}{2}, p \in \mathbb{Z} \rangle$  is a nilpotent Lie algebra – extending that of  $\mathfrak{sch}_1$ . Looking at  $\mathfrak{sv}$  from the point of view of the theory of representations of the Virasoro algebra (also called for short: *vir-modules*) proved to be a very fruitful approach (see [15] for the following exposition). The Virasoro algebra has a class of representations  $\mathcal{F}_\lambda$  called *tensor-density modules*; formally,  $\mathcal{F}_\lambda \cong \langle f(z)dz^{-\lambda} \rangle$  with action  $[L_n, z^m dz^{-\lambda}] = (\lambda n - m)z^{n+m} dz^{-\lambda}$ , allowing an identification with a primary field of conformal weight  $(1 + \lambda)$ . Then  $\mathfrak{h} \cong \mathcal{F}_{-\frac{1}{2}} \oplus \mathcal{F}_0$  as a  $\mathfrak{g}$ -module.

The following theorem classifies *deformations* of the Lie algebra  $\mathfrak{sv}$  and central extensions of these. Central extensions are familiar to anybody acquainted with conformal field theory; they

are an essential tool to construct covariant *quantum* theories. On the other hand, deformations (in this context) answer the question : how 'general' is  $\mathfrak{sv}$  ? Is it the only Lie algebra that enjoys more or less the same structure as  $\mathfrak{sv}$ , or does it belong to a family? The Virasoro algebra, which is *simple*, is known to have no non-trivial deformations; on the contrary, it is mathematically natural to try deforming Lie algebras that enjoy a semi-direct product structure; therefore, the problem of finding deformations of  $\mathfrak{sv}$  is *a priori* quite sensible. The answer combines *all* anisotropy exponents into a single structure, making it plausible to find general theories for anisotropic statistical physics not depending on any particular exponent.

**Theorem 1** (see [15], section 5)

- (i) *The most general deformation of the Lie algebra  $\mathfrak{sv}$  is given by  $\mathfrak{sv}_{\lambda,\mu,\nu}$  with Lie bracket  $[X, Y]_{\lambda,\mu,\nu} = [X, Y]_{\mathfrak{sv}} + \lambda c_1(X, Y) + \mu c_2(X, Y) + \nu c_3(X, Y)$ , where  $c_1(L_n, Y_m) = \frac{n}{2} Y_{n+m}$ ,  $c_1(L_n, M_m) = n M_{n+m}$ ;  $c_2(L_n, Y_m) = -Y_{n+m}$ ,  $c_2(L_n, M_m) = -2M_{n+m}$ ;  $c_3(L_n, L_m) = (m - n)M_{n+m}$  (all missing components vanishing).*
- (ii) *The Lie algebra  $\mathfrak{sv}_\lambda := \mathfrak{sv}_{\lambda,0,0}$  has exactly one family of central extensions, given by the extension by zero of the Virasoro cocycle,  $(\mathfrak{sv}_\lambda)^k \cong \mathfrak{vir}^k \ltimes \mathfrak{h}_\lambda$ , except for  $\lambda = 1, -3$ , in which cases the cocycle  $c : (Y_p, Y_q) \rightarrow p^3 \delta_{p+q}^0$ ,  $(L_p, M_q) \rightarrow -p^3 \delta_{p+q}^0$  ( $\lambda = 1$ ),  $c' : (Y_p, Y_q) \rightarrow \frac{1}{q} \delta_{p+q}^0$  ( $\lambda = -3$ ) defines another independent family (all missing components vanishing).*
- (iii) *The Lie algebra  $\mathfrak{tsv}_\lambda \cong \mathfrak{sv}_{\lambda, \frac{1}{2}, 0}$  – for which we choose to give integer indices to the components of  $Y$  by shifting them by  $\frac{1}{2}$ , see comments below – has no other central extensions than those given by the extension by zero of the Virasoro cocycle, except for the cocycles*

$$c_{-3} : (L_n, Y_m) \rightarrow \delta_{n+m}^0 \quad (\lambda = -3), \quad (7)$$

$$c_{-1} : (L_n, Y_m) \rightarrow n^2 \delta_{n+m}^0 \quad (\lambda = -1), \quad (8)$$

$$c_1 : (L_n, Y_m) \rightarrow n^3 \delta_{n+m}^0; \quad c'_1 : (L_n, M_m) \rightarrow n^3 \delta_{n+m}^0, \quad (Y_n, Y_m) \rightarrow n^3 \delta_{n+m}^0 \quad (\lambda = 1). \quad (9)$$

The proof is long and technical and relies heavily on the cohomological machinery for Lie algebras of vector fields developed by Fuks (see [3] or [5]). Note the very different results for the central extensions of  $\mathfrak{sv}_\lambda$  and  $\mathfrak{tsv}_\lambda$ ; this follows immediately from an easy lemma which states that any cocycle  $c$  generating a central extension may be chosen to satisfy  $c(X_n, X_m) = 0$  ( $X = L, Y$  or  $M$ ) if  $n + m \neq 0$ .

While  $c_2$  amounts to a simple shift in the indices of  $Y, M$  (similarly to the Ramond/Neveu-Schwarz sectors in supersymmetric conformal field theory), yielding in particular the *twisted Schrödinger-Virasoro algebra*  $\mathfrak{tsv}$  (with same Lie bracket as  $\mathfrak{sv}$  but *integer indices* for the  $Y$  generators), and the signification of  $c_3$  is unclear as yet,  $c_1$  corresponds to a shift in the anisotropy exponent  $z$ . Namely, it gives rise to a family of Lie algebras  $\mathfrak{sv}_\lambda \cong \mathfrak{g} \ltimes \mathfrak{h}_\lambda$ , with  $\mathfrak{h}_\lambda \cong \mathcal{F}_{\frac{1+\lambda}{2}} \oplus \mathcal{F}_\lambda$  as  $\mathfrak{g}$ -module. It can be proved (see [15], section 3) that

$$\begin{aligned} L_n &= -t^{n+1} \partial_t - \frac{1+\lambda}{2} (n+1) t^n r \partial_r - \left[ \lambda (n+1) t^n \zeta + \frac{1+\lambda}{4} (n+1) n t^{n-1} r^2 \right] \partial_\zeta - x(n+1) t^n \\ Y_m &= -t^{m+\frac{1+\lambda}{2}} \partial_r - (m + \frac{1+\lambda}{2}) t^{m+\frac{\lambda-1}{2}} r \partial_\zeta, \quad M_p = -t^{p+\lambda} \partial_\zeta \end{aligned} \quad (10)$$

gives for any *scaling exponent*  $x$  an explicit realization of  $\mathfrak{sv}_{\lambda, \frac{\lambda}{2}, 0}$ , generalizing the original realization of  $\mathfrak{sv}$  introduced in [6] whose restriction to  $\mathfrak{sch}_1$  coincides with that given in the

introduction; in particular, the explicit form  $L_0 = -t\partial_t - (\frac{1}{2} + \frac{\lambda}{2})r\partial_r - \lambda\zeta\partial_\zeta + x$  shows the connection with dynamical scaling with  $z = \frac{2}{1+\lambda}$ , while carrying a third coordinate  $\zeta$  as in (2). The Lie algebras  $\mathfrak{sv}_{\lambda, \frac{\lambda}{2}, 0}$  admit for integer  $\lambda = 0, 1, \dots$  finite-dimensional Lie subalgebras  $\mathfrak{s}_\lambda = \langle L_{-1}, L_0, L_1 \rangle \ltimes \langle Y_{-\frac{1+\lambda}{2}}, \dots, Y_{\frac{1+\lambda}{2}}; M_{-\lambda}, \dots, M_\lambda \rangle$ . Note that the  $Y$ -generators of  $\mathfrak{sv}_{\lambda, \frac{\lambda}{2}, 0}$ , hence of  $\mathfrak{s}_\lambda$ , bear *integer* indices when  $\lambda$  is even and *half-integer indices* when  $\lambda$  is odd.

### Theorem 2.

- (i) Let  $\lambda = 0, 1, \dots$ . Set  $t = t_1 - t_2$ ,  $r = r_1 - r_2$ ,  $\zeta = \zeta_1 - \zeta_2$ . The two-point functions  $C(t_1, r_1, \zeta_1; t_2, r_2, \zeta_2) = \langle \Phi_1(t_1, r_1, \zeta_1) \Phi_2(t_2, r_2, \zeta_2) \rangle$  of fields  $\Phi_{1,2}$  covariant under the realization (10) of  $\mathfrak{s}_\lambda$ , with respective scaling exponents  $x_1, x_2$ , are equal (up to a constant) to  $\delta_{x_1, x_2} t^{-2x_1}$ , except for  $\lambda = 0$ : then the general two-point function is equal to  $\delta_{x_1, x_2} t^{-2x_1} f(\frac{r^2}{2t} - \zeta)$ , where  $f$  is an arbitrary scaling function.
- (ii) Let  $\lambda = 1, 2, \dots$ . Then the three-point functions  $C(t_1, r_1, \zeta_1; t_2, r_2, \zeta_2; t_3, r_3, \zeta_3)$  (same notations) are given up to a constant by the usual conformal Ansatz  $(t_1 - t_2)^{-x_1 - x_2 + x_3} (t_2 - t_3)^{-x_2 - x_3 + x_1} (t_3 - t_1)^{-x_3 - x_1 + x_2}$ .

The  $\mathfrak{s}_0$ -case is already known (see [8]) since  $\mathfrak{s}_0 \cong \mathfrak{sch}_1$ ; the scaling function for the two-point functions gives (after inverting the Fourier transform in the mass, see introduction) the exponential in the heat kernel  $\exp\left(-\frac{\mathcal{M}r^2}{2t}\right)$ . Apparently, covariance constraints are too strong for  $\lambda > 0$  to give anything else than the rather trivial, space-independent conformally invariant two- or three-point functions in the variable  $t$ . Yet one should not throw away these ideas too quickly; we conjecture that, by adapting formulas (12) below to  $\mathfrak{sv}_\varepsilon$ , one can find *vector*  $\mathfrak{s}_\lambda$ -covariant fields for which the two- and three-point functions mix the coordinates  $t, r, \zeta$ . One should maybe find physical equations that are invariant under these representations for a start. These preliminary results look very different from those of [7].

**2.** The Lie algebra  $\mathfrak{sv}$  has several graduations; one of them is given by the adjoint action of the full dilation generator  $t\partial_t + r\partial_r + \zeta\partial_\zeta$  in the realization (10), yielding  $\mathfrak{sv}^{(n)} = \langle L_n, Y_{n+\frac{1}{2}}, M_{n+1} \rangle$ . The Lie subalgebra  $\mathfrak{sv}^{(0)}$  is solvable, it admits a simple class of finite-dimensional representations given by

$$\rho(L_0) = \sigma\left(\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\right), \quad \rho(Y_{\frac{1}{2}}) = \sigma\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), \quad \rho(M_1) = \rho(Y_{\frac{1}{2}})^2 \quad (11)$$

where  $\sigma$  is a spin- $s$  representation of  $\mathfrak{sl}(2, \mathbb{R})$  on a space  $\mathcal{H}_\rho$ . Now  $\rho$  gives rise by the *coinduction method* to a representation  $\tilde{\rho}$  of  $\mathfrak{sv}$  on the space of  $\mathcal{H}_\rho$ -valued functions of three variables  $t, r, \zeta$  defined by (see [15], Theorem 4.2)

$$\begin{aligned} \tilde{\rho}(L_n) &= \left( -t^{n+1}\partial_t - \frac{1}{2}(n+1)t^n r\partial_r - \frac{1}{4}(n+1)nt^{n-1}r^2\partial_\zeta \right) \otimes \text{Id}_{\mathcal{H}_\rho} \\ &+ (n+1)t^n \rho(L_0) + \frac{1}{2}(n+1)nt^{n-1}r \rho(Y_{\frac{1}{2}}) + \frac{1}{4}(n+1)n(n-1)t^{n-2}r^2 \rho(M_1) \\ \tilde{\rho}(Y_m) &= \left( -t^{m+\frac{1}{2}}\partial_r - (m+\frac{1}{2})t^{m-\frac{1}{2}}r\partial_\zeta \right) \otimes \text{Id}_{\mathcal{H}_\rho} + (m+\frac{1}{2})t^{m-\frac{1}{2}}\rho(Y_{\frac{1}{2}}) + (m^2 - \frac{1}{4})t^{m-\frac{3}{2}}r \rho(M_1) \\ \tilde{\rho}(M_p) &= -t^p\partial_\zeta \otimes \text{Id}_{\mathcal{H}_\rho} + pt^{p-1} \rho(M_1). \end{aligned} \quad (12)$$

This family of representations is very rich (see [15], section 4.2 for a list of physical examples); it is the exact mathematical analogue of the primary fields or otherwise tensor density modules  $\mathcal{F}_\lambda$  for  $\mathfrak{sv}$ , while the most naive approach by induction, leading to unitary highest weight representations for  $\mathfrak{vir}$ , seems to fail here.

**3.** In a work in progress (see [17]), we construct representations of  $\mathfrak{sv}$  by means of current algebras: that is,

$$L(t) = \sum_{n \in \mathbb{Z}} L_n t^{-n-2}, \quad Y(t) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} Y_n t^{-n-\frac{3}{2}}, \quad M(t) = \sum_{n \in \mathbb{Z}} M_n t^{-n-1}$$

are constructed as polynomials in Kac-Moody current fields satisfying the non-trivial operator product expansions

$$L(t_1)L(t_2) \sim \frac{\partial L(t_1)}{t_1 - t_2} + \frac{2L(t_2)}{(t_1 - t_2)^2} + \frac{k/2}{(t_1 - t_2)^4}, \quad k \in \mathbb{R} \quad (13)$$

$$L(t_1)Y(t_2) \sim \frac{\partial Y(t_2)}{t_1 - t_2} + \frac{\frac{3}{2}Y(t_2)}{(t_1 - t_2)^2}, \quad (14)$$

$$L(t_1)M(t_2) \sim \frac{\partial M(t_2)}{t_1 - t_2} + \frac{M(t_2)}{(t_1 - t_2)^2} \quad (15)$$

$$Y(t_1)Y(t_2) \sim \frac{\partial M(t_2)}{t_1 - t_2} + \frac{2M(t_2)}{(t_1 - t_2)^2} \quad (16)$$

In analogy with conformal field theory, a  $\rho$ -Schrödinger conformal field  $\Phi$  is defined as a (formal) infinite series  $\Phi(t, r, \zeta) = \left( \sum_{\xi \in \mathbb{Z}} \Phi^{a, \xi}(t, \zeta) r^\xi \right) e_a$ , ( $e_a$ ) basis of  $\mathcal{H}_\rho$ , where the  $\Phi^{a, \xi}(t, \zeta)$  are mutually local fields with respect to the time variable  $t$  – which are also mutually local with the  $\mathfrak{sv}$ -fields  $L(t), Y(t), M(t)$  – with the following properties:

$$\begin{aligned} L(t_1)\Phi^{a, \xi}(t_2, \zeta) &\sim \frac{\partial \Phi^{a, \xi}(t_2, \zeta)}{t_1 - t_2} + \frac{(\frac{1}{2}\xi)\Phi^{a, \xi}(t_2, \zeta) - \rho(L_0)_b^a \Phi^{b, \xi}(t_2, \zeta)}{(t_1 - t_2)^2} \\ &+ \frac{\frac{1}{2}\partial_\zeta \Phi^{a, \xi-2}(t_2, \zeta) - \rho(Y_{\frac{1}{2}})_b^a \Phi^{b, \xi-1}(t_2, \zeta)}{(t_1 - t_2)^3} - \frac{\frac{3}{2}\rho(M_1)_b^a \Phi^{b, \xi-2}(t_2, \zeta)}{(t_1 - t_2)^4} \\ Y(t_1)\Phi^{a, \xi}(t_2, \zeta) &\sim \frac{(1 + \xi)\Phi^{a, \xi+1}(t_2, \zeta)}{t_1 - t_2} + \frac{\partial_\zeta \Phi^{a, \xi-1}(t_2, \zeta) - \rho(Y_{\frac{1}{2}})_b^a \Phi^{b, \xi}(t_2, \zeta)}{(t_1 - t_2)^2} - \frac{2\rho(M_1)_b^a \Phi^{b, \xi-1}(t_2, \zeta)}{(t_1 - t_2)^3} \\ M(t_1)\Phi^{a, \xi}(t_2, \zeta) &\sim \frac{\partial_\zeta \Phi^{a, \xi}(t_2, \zeta)}{t_1 - t_2} - \frac{\rho(M_1)_b^a \Phi^{b, \xi}(t_2, \zeta)}{(t_1 - t_2)^2} \end{aligned} \quad (17)$$

See [17] for constructions using  $U(1)$ -currents or more general Kac-Moody currents.

As a first application, we now give a surprising relationship with a Dirac-type equation.

### Theorem 3.

(i) The space of spinor solutions  $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$  of the constrained 3D-Dirac equation

$$\partial_r \phi = \partial_t \psi, \quad \partial_r \psi = \partial_\zeta \phi, \quad \partial_\zeta \psi = 0 \quad (18)$$

is in one-to-one correspondence with the space of triples  $(h_0^-, h_0^+, h_1)$  of functions of  $t$  only: a natural bijection may be obtained by setting

$$\phi(t, r, \zeta) = (h_0^-(t) + \zeta h_0^+(t)) + r h_1(t) + \frac{r^2}{2} \partial h_0^+(t), \quad \psi(t, r, \zeta) = \int_0^t h_1(u) du + r h_0^+(t). \quad (19)$$

(ii) Put

$$\Phi(t, r, \zeta) = (\bar{b}^-(t) + \zeta \bar{b}_0^+(t)) + r a(t) + \frac{r^2}{2} \partial \bar{b}^+(t), \quad \Psi(t, r, \zeta) = \int_0^t a(u) du + r \bar{b}^+(t) \quad (20)$$

where  $a$  is a free boson and  $\bar{b}^\pm$  are charge-conjugate superbosons, see for instance [12], by which we mean that the following operator product expansions hold:

$$a(t_1)a(t_2) \sim \frac{1}{(t_1 - t_2)^2}, \quad \bar{b}^\pm(t_1)\bar{b}^\mp(t_2) \sim \pm \frac{1}{t_1 - t_2}, \quad a(t_1)\bar{b}^\pm(t_2) \sim 0 \quad (21)$$

Then  $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$  is a  $\rho$ -Schrödinger-conformal field for the  $\mathfrak{sv}$ -fields

$$L(t) = \frac{1}{2} : a^2 : (t) + \frac{1}{2} (:\bar{b}^+ \partial \bar{b}^- : (t) - :\bar{b}^- \partial \bar{b}^+ : (t)), \quad Y(t) = : a \bar{b}^+ :, \quad M = \frac{1}{2} : (\bar{b}^+)^2 :, \quad (22)$$

$\rho$  being the two-dimensional character defined by

$$\rho(L_0) = \begin{pmatrix} -\frac{1}{2} & \\ & 0 \end{pmatrix}, \quad \rho(Y_{\frac{1}{2}}) = \rho(M_1) = 0. \quad (23)$$

This *constrained* Dirac equation also appears in [15], see section 3.5 for details, in connection with a family of wave equations indexed by the dimension that are invariant under the Lie algebras  $\mathfrak{sv}_\lambda$  or related Lie algebras of the same type  $\mathfrak{vir} \ltimes \mathfrak{h}$ ,  $\mathfrak{h}$  infinite-dimensional nilpotent.

4. Let us finally come back to what we said in the introduction, namely that  $\mathfrak{sv}$  is *one* of the substitutes for  $\mathfrak{vir}$  in  $z = 2$  non-equilibrium dynamics. The Schrödinger-Virasoro algebra actually appears to be a quotient of the *twisted Poisson algebra*  $\tilde{\mathcal{P}}^{(2)}$  [9], defined as the associative algebra of functions  $f(p, q) := \sum_{i \in \frac{1}{2}\mathbb{Z}} \sum_{j \in \mathbb{Z}} c_{i,j} p^i q^j$  with usual multiplication and Lie-Poisson bracket defined by  $\{f, g\} := \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$ . Namely, if we define a graduation  $\text{gra} : \tilde{\mathcal{P}}^{(2)} \rightarrow \{0, \frac{1}{2}, 1, \dots\}$  on the associative algebra  $\tilde{\mathcal{P}}^{(2)}$  by setting  $\text{gra}(q^n p^m) := m$ , and call  $\tilde{\mathcal{P}}_{\leq \kappa}^{(2)}$ ,  $\kappa \in \frac{1}{2}\mathbb{Z}$  the vector subspace of  $\tilde{\mathcal{P}}^{(2)}$  consisting of all elements with graduation  $\leq \kappa$ , then  $\mathfrak{sv} \cong \tilde{\mathcal{P}}_{\leq 1}^{(2)} / \tilde{\mathcal{P}}_{\leq -\frac{1}{2}}^{(2)}$ . This construction has a rich family of supersymmetric analogues, see again [9].

The interesting point for this short article is that this isn't mere mathematical fancy. Namely, one has

**Theorem 4.**

*The free diffusion equation in one space dimension  $(2\mathcal{M}\partial_t - \partial_r^2)\Phi(t, r) = 0$  is invariant under the family of integro-differential operators  $f(t + \mathcal{M}r\partial_r^{-1})\partial_r^{2\kappa}$ , where  $f$  is any function in one variable and  $\kappa \in \frac{1}{2}\mathbb{Z}$ .*

Through the transformation  $\Phi \mapsto \tilde{\Phi}$  defined as  $\tilde{\Phi}(t, \xi) = \int_{\mathbb{R}} \frac{e^{-\mathcal{M}r^2/2\xi}}{\sqrt{\xi}} \Phi(t, r) dr$ , the operator  $t + \mathcal{M}r\partial_r^{-1}$  is seen to go over to  $t + \xi$  and  $\frac{\partial^2}{2\mathcal{M}}\partial_r^{-1}$  to  $\partial_\xi$ . In the 'light-cone' coordinates  $z = t + \xi$ ,

$\bar{z} = t - \xi$ , this family of integro-differential operators becomes the family of fractional differential operators  $f(z)(\partial_z - \partial_{\bar{z}})^\kappa$ , which (restricting to values  $\kappa \leq 1$ ) is easily seen to be isomorphic to  $\tilde{\mathcal{P}}_{\leq 1}^{(2)}$ . The vector fields  $z^{n+1}\partial_z + \frac{n+1}{4}z^n$ , when rewritten in the original coordinates  $(t, r)$ , have (up to the equations of motion) the same *differential part* as the Virasoro generators of the Schrödinger-Virasoro algebra  $\mathfrak{sv}$ , in its realization (10). This fact points to an unexpected direct relation between ( $z = 2$ ) non-equilibrium dynamics and conformal field theory that is undoubtedly very promising.

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